SOLUTION OF MASSLESS SPIN ONE WAVE EQUATION IN ROBERTSON-WALKER SPACE-TIME

YUSUF SUCU [1] and NURI UNAL[2]

Akdeniz University, Department of Physics, PK 510, 07058 Antalya, Turkey

Abstract

We generalize the quantum spinor wave equation for photon into the curved space-time and discuss the solutions of this equation in Robertson-Walker space-time and compare them with the solution of the Maxwell equations in the same space-time.

PACS numbers: 03.65.Pm, 98. 80. Cq

1   Introduction

In 1957 Brill and Wheeler investigated the solution of the Dirac equation in Schwarzschild metric and they also discussed the geodesic equation for the photon in the same metric [1]. Later there are a lot of studies on the solution of the Klein-Gordon equation in expanding universes. To understand the contributions of the spin to particle creation and the vacuum structure of the curved space-times some authors discussed the solution of the Dirac equation in Robertson-Walker (RW) metrics [2]-[5].

Generally the photons are formulated as the quantum of the Maxwell fields in flat and curved space-times. There are continuous attempts to write the Maxwell equations as the spinor equations and also to represent the photon by a quantum wave equation for photon [6]-[13]. All massless spinning particle wave equations can be written as the spinor equations in the Newmann-Penrose formalism[14].

---

[1]E-mail : ysucu@sci.akdeniz.edu.tr

[2]E-mail : unal@sci.akdeniz.edu.tr

arXiv:gr-qc/0205097v2 8 Apr 2005

The spin is introduced in Dirac equation and there are a lot of models to discuss the the classical analogue of this new degree of freedom[15]. In 1984, Barut and Zanghi proposed a classical model of the zitterbewegung and discussed the free particle trajectories in this model[16]. The quantization of this model is discussed from different aspects[17] and it gives the Dirac equation for the spin-1/2 electron, the Duffin-Kemer-Petiau equation for the spin-1 particle as well as the higher-spinwave equations[18].

Recently one of us proposed a classical model of the zitterbewegung for the massless spinning particles[19] and the quantization of this model gives the spinor wave equations for the massless spinning particle as well as two component neutrino equation.

The aim of this paper is to discuss the generalization of the classical zitterbewegung model, to obtain the quantum wave equations for photons in curved background metric, and as an example, to discuss the solutions of this wave equation in the RW metrics.

In section two we discuss the generalization of the simple model of the zitterbewegung to the curved space-times and the quantization of this classical system and derive the Weyl equation and spin-1 analogue of it in the curved space-times. In section four we discuss the solution of this spin-1 wave equation for the Robertson- Walker space-times. For the comparison we also obtain the solutions of the Maxwell equations for the same metrics in Appendix.

2   Classical and quantum system

The classical model of the zitterbewegung for electron is formulated in the configuration space $M^4 \otimes C^4$ with the canonically conjugate internal and external coordinates and momenta of the particle:(x, p) and $\left(\eta^\dagger, \eta\right)$. Here $\eta^\dagger$ and η are the 4 component complex spinors.For the massless particles the action of the classical spinning particle in curved space-time is

$$A = \int ds \left[\frac{1}{2i} \left(\frac{d\eta^\dagger}{ds}\eta - \eta^\dagger \frac{d\eta}{ds} \right) + p_\mu \frac{dx^\mu}{ds} - \eta^\dagger \sigma^\mu \eta \left(p_\mu + i\eta^\dagger \Gamma_\mu \eta \right) \right], \quad (1)$$

where σ^α is 2×2 Pauli matrices, $\eta(s)$ is a two component complex spinor, Γ_μ is the spin connection and given in next section p^μ and x^μ are the canonical conjugate, external coordinates of the particle. The internal dynamics of the particle is described by the $\eta^\dagger$ and η, which are the canonical conjugate,

holomorphic coordinates. In the second term of the Lagrangian the internal and external dynamics are coupled by the Lagrange multiplier p. The internal coordinates describe the four coupled oscillators. We choose $\hbar = c = 1$. The classical equations of motion for x^μ, p^μ, $\eta^\dagger$ and η are derived in Reference [19].

The elements of the configuration space of the spinning particle are x^μ and $\eta^\dagger$. Then the Schrödinger wave function is $\Phi\left(x, \eta^\dagger; s\right)$ and it satisfies the following wave equation:

$$i\frac{\partial}{\partial s}\Phi\left(x, \eta^\dagger; s\right) = \widehat{H}\Phi\left(x, \eta^\dagger; s\right),\tag{2}$$

where $\widehat{H}$ is the Hamiltonian and given by

$$\widehat{H} = \widehat{\eta^\dagger}\gamma^\mu\widehat{\eta}\left(\widehat{p}_\mu + i\widehat{\eta^\dagger}\Gamma_\mu\widehat{\eta}\right).\tag{3}$$

$\widehat{H}$ is the function of the canonical conjugate variables $\widehat{\eta^\dagger}$ and $\widehat{\eta}$ and $\widehat{p}^\mu$ and $\widehat{x}^\mu$. $\widehat{\eta}$ and $\widehat{p}^\mu$ are represented as

$$\widehat{\eta} = \frac{\partial}{\partial\eta^\dagger},$$

$$\widehat{p}^\mu = i\frac{\partial}{\partial x_\mu}.\tag{4}$$

$\Phi\left(x, \eta^\dagger; s\right)$ represents all eigenvalues of the spin. To obtain the wave equations for certain eigenvalues of the spin we expand Φ as a power series of $\eta^\dagger$:

$$\Phi\left(x, \eta^\dagger; s\right) = \Psi_0\left(x, s\right) + \eta_\alpha^\dagger\Psi_\alpha\left(x, s\right) + \frac{1}{2}\left(\eta_\alpha^\dagger\eta_\beta^\dagger + \eta_\beta^\dagger\eta_\alpha^\dagger\right)\Psi_{\alpha\beta}\left(x, s\right) + \ldots\tag{5}$$

In this expansion $\Psi_{\alpha\beta\gamma\ldots}$ is symmetric under the exchange of the indices and thus for example $\Psi_{\alpha\beta}$ has 3 independent components. We substitute this expansion into the Schrödinger equation (2). Then each power of $\eta^\dagger$ satisfies the separate wave equations:

$$\left[i\frac{\partial}{\partial s} - \frac{1}{2\Lambda}p_\mu g^{\mu v}p_v\right]\Psi_\circ\left(x, s\right) = 0,$$

$$\left[i\frac{\partial}{\partial s} - \sigma^\mu\left(\widehat{p}_\mu - i\Gamma_\mu\right)\right]_{\alpha\beta}\Psi_{\alpha\beta}\left(x, s\right) = 0,\tag{6}$$

3

$$\{i\frac{\partial}{\partial s}I \otimes I - \Sigma^\mu(x)\left[p_\mu + i\Gamma_\mu(x) \otimes I + I \otimes i\Gamma_\mu(x)\right]\}_{\alpha_1\alpha_2,\beta_1\beta_2}\Psi_{\beta_1\beta_2} = 0.$$

The first, second and third of these equations are the Klein Gordon equation with a parameter Λ, the neutrino equation and the photon wave equations in general coordinate frame respectively and the $\Sigma^\mu(x)$ in the third equation is

$$\Sigma^\mu(x) = \sigma^\mu(x) \otimes I + I \otimes \sigma^\mu(x). \tag{7}$$

In flat Minkowski space-time the photon wave equation becomes

$$\{i\frac{\partial}{\partial s}I \otimes I - \widehat{p}_\mu(\sigma^\mu \otimes I + I \otimes \sigma^\mu)\}_{\alpha_1\alpha_2,\beta_1\beta_2}\Psi_{\beta_1\beta_2} = 0. \tag{8}$$

For the massless particles $\frac{\partial}{\partial s} = 0$ and Eq.(7) becomes

$$2iI \otimes I\frac{\partial\Psi}{\partial t} = \widehat{\vec{p}}\left[\vec{\sigma} \otimes I + I \otimes \vec{\sigma}\right]\Psi = 0. \tag{9}$$

In equation (9) Ψ has three components. We write this equation explicitly by introducing a vector wave function $\vec{\Psi}$ such that

$$\vec{\Psi} = \left(\Psi_-, \sqrt{2}\Psi_0, \Psi_+\right), \tag{10}$$

where $\Psi_{-,+,0}$ are the components of the spinor Ψ. Then Eq.(9) becomes

$$i\frac{\partial\vec{\Psi}}{\partial t} = \nabla \times \vec{\Psi}.$$

This is the six of the Maxwell equations and if $\vec{\Psi}$ is time dependent, then also $\nabla\vec{\Psi} = 0$. Thus the spinor equation (6) (for photon) which is obtained from the generalization of the Eq.(9) to curved space-time is equivalent to the generalization of Maxwell equations to curved space-time. Since this equation has only first order derivatives with respect to space and times we define $\Psi^\dagger\sigma^\mu\Psi$ as the probability current for the photon.

3 Solution of the photon wave equation in RW metrics

The RW metric is

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = dt^2 - a^2(t)\left(\frac{dr^2}{1 - kr^2} + r^2d\theta^2 + r^2\sin^2\theta d\varphi^2\right), \tag{11}$$

where $a^2(t)$ is the expansion parameter and k is the curvature parameter between -1 and 1. The metric tensor can be represented in terms of the symmetric form of the vierbein L_μ^α

$$g_{\mu\nu} = L_\mu^\alpha L_\nu^\beta \eta_{\alpha\beta}, \tag{12}$$

where $\eta_{\alpha\beta}$ is metric of the Minkowski frame and $\eta_{\alpha\beta} = (+1, -1, -1, -1)$. The vierbein of the metric is

$$L_\circ^\circ = 1, \quad L_\circ^i = 0$$

and

$$L_j^i = L_i^j = \frac{a}{r}\delta_{ij} + a\frac{(1-\rho)}{\rho r^2}x^i x^j, \tag{13}$$

where ρ is $(1 - kr^2)^{\frac{1}{2}}$. Then the Dirac matrices are related to each other by

$$\sigma^\mu(x) = L_\alpha^\mu(x)\sigma^\alpha = g^{\mu\nu}L_\nu^\alpha \eta_{\alpha\beta}\sigma^\beta$$

and

$$\sigma_\mu(x) = L_\mu^\alpha(x)\sigma_\alpha, \tag{14}$$

where $\sigma^\mu(x)$ and σ^α are Pauli matrices in the general coordinate frame and the Minkowski frame respectively and they satisfy the following anticommutation relations:

$$\sigma^\alpha \sigma^\beta + \sigma^\beta \sigma^\alpha = 2\eta^{\alpha\beta},$$

$$\sigma^\alpha(x)\sigma^\beta(x) + \sigma^\beta(x)\sigma^\alpha(x) = 2g^{\alpha\beta}(x). \tag{15}$$

By using the expression of $L_\alpha^\mu(x)$ we write $\sigma^\mu(x)$ as

$$\sigma^\circ(x) = \sigma^\circ,$$

$$\overrightarrow{\sigma}(x) = -\frac{1}{a}\left[\overrightarrow{\sigma} - \frac{1-\rho}{r^2}\overrightarrow{r}\,\overrightarrow{r}\cdot\overrightarrow{\sigma}\right]. \tag{16}$$

The spin connection is derived by the following expression:

$$\Gamma_\mu(x) = -\frac{1}{8}\left[\sigma^\nu(x), \sigma_{\nu;\mu}(x)\right].$$

5

It is

$$\Gamma^{\circ} = 0$$

$$\overrightarrow{\Gamma} = -\frac{1}{2}\left[\dot{a}\left(\sigma^{\circ}\overrightarrow{\sigma} + \frac{1-\rho}{\rho r^2}\overrightarrow{r}\sigma^{\circ}\overrightarrow{r}\cdot\overrightarrow{\sigma}\right) - i\frac{(1-\rho)}{r^2}\left(\overrightarrow{\sigma}\times\overrightarrow{r}\right)\right]. \qquad (17)$$

Then the interaction term becomes

$$-[\overrightarrow{\sigma}(x)\otimes \mathrm{I} + \mathrm{I}\otimes\overrightarrow{\sigma}(x)]\cdot\left[\overrightarrow{\Gamma}(x)\otimes \mathrm{I} + \mathrm{I}\otimes\overrightarrow{\Gamma}(x)\right] =$$

$$-\frac{1}{2a}\{[\overrightarrow{\sigma} - (1-\rho)\overrightarrow{r}\overrightarrow{\sigma}\cdot\overrightarrow{r}]\otimes \mathrm{I} + \mathrm{I}\otimes[\overrightarrow{\sigma} - (1-\rho)\overrightarrow{r}\overrightarrow{\sigma}\cdot\overrightarrow{r}]\}$$

$$\times\{\left[\dot{a}\left(\overrightarrow{\sigma} + \frac{1-\rho}{\rho r^2}\overrightarrow{r}\overrightarrow{\sigma}\cdot\overrightarrow{r}\right) - i\frac{1-\rho}{r^2}\left(\overrightarrow{\sigma}\times\overrightarrow{r}\right)\right]\otimes \mathrm{I}$$

$$+\mathrm{I}\otimes\left[\dot{a}\left(\overrightarrow{\sigma} + \frac{1-\rho}{\rho r^2}\overrightarrow{r}\overrightarrow{\sigma}\cdot\overrightarrow{r}\right) - i\frac{1-\rho}{r^2}\left(\overrightarrow{\sigma}\times\overrightarrow{r}\right)\right]\}. \qquad (18)$$

We will integrate the time and the angular coordinates of the wave function by using the group theoretical methods[20]. For this reason we derive the photon wave equation from the variation of the following action with respect to $\Phi^{\dagger}\left(\overrightarrow{r},t\right)$:

$$A = \int d^4x\sqrt{-g}\Phi^{\dagger}\left(\overrightarrow{x},t\right)\{2i\mathrm{I}\otimes \mathrm{I}\frac{\partial}{\partial t}$$

$$-\overrightarrow{\Sigma}(x)\left[\overrightarrow{p} + i\overrightarrow{\Gamma}(x)\otimes \mathrm{I} + \mathrm{I}\otimes i\overrightarrow{\Gamma}(x)\right]\}\Phi\left(\overrightarrow{x},t\right). \qquad (19)$$

Because of the spherical symmetry of the intersection terms we write the action in spherical coordinates (r,θ,φ). It is

$$A = \int dt dr d\theta d\varphi\sqrt{-g}\Phi^{\dagger}[2i\mathrm{I}\otimes \mathrm{I}\frac{\partial}{\partial t} + i\frac{\dot{a}}{a}\left(\Sigma^r\Sigma^r + \Sigma^{\theta}\Sigma^{\theta} + \Sigma^{\varphi}\Sigma^{\varphi}\right)$$

$$+\frac{i}{a}\left(\rho\Sigma^r\frac{\partial}{\partial r} + \frac{1}{r}\Sigma^{\theta}\frac{\partial}{\partial\theta} + \frac{1}{r\sin\theta}\Sigma^{\varphi}\frac{\partial}{\partial\varphi}\right) - i\frac{(1-\rho)}{a}\frac{\Sigma^r}{r}]\Phi, \qquad (20)$$

where $\dot{a} = \frac{da}{dt}$, $\sqrt{-g}$ is

$$\sqrt{-g} = \frac{a^3}{1-kr^2}r^2\sin\theta, \qquad (21)$$

6

and Σ^r, Σ^θ and Σ^φ are the components of the $\overrightarrow{\Sigma}$ along the axis r, θ and φ. We rewrite the action in equation(20) as

$$A = i \int d\tau d\chi d\theta d\varphi e^{\frac{3\alpha}{2}} \xi^2 (\chi) \sin \theta \Phi^\dagger (\chi, \theta, \varphi; \tau) \left[2\mathrm{I} \otimes \mathrm{I} \left(\frac{\partial}{\partial \tau} + \frac{d\alpha}{d\tau} \right) + \right.$$

$$\left. \Sigma^\chi \left(\frac{\partial}{\partial \chi} + \frac{\xi' (\chi) - 1}{\xi (\chi)} \right) + \frac{1}{\xi (\chi)} \left(\Sigma^\theta \frac{\partial}{\partial \theta} + \frac{1}{\sin \theta} \Sigma^\varphi \frac{\partial}{\partial \varphi} \right) \right] \Phi (\chi, \theta, \varphi; \tau), \quad (22)$$

where the new coordinates τ and χ are defined as

$$d\tau = e^{\frac{\alpha(\tau)}{2}} dt = a (t) dt, \ r = \xi (\chi) = \frac{1}{\sqrt{k}} \sin \sqrt{k} \chi \quad \text{and} \quad \xi' (\chi) = \frac{d\xi}{d\chi}. \quad (23)$$

Then the metric becomes

$$ds^2 = e^{\alpha(\tau)} \left[d\tau^2 - d\chi^2 - \xi^2 (\chi) \left(d\theta^2 + \sin^2 \theta \ d\varphi^2 \right) \right]. \quad (24)$$

To eliminate the τ dependent potentials we introduce a new function f such that

$$\Phi (\chi, \theta, \varphi, \tau) = \xi^{-1} (\chi) \ e^{-\alpha} f (\chi, \theta, \varphi; \tau). \quad (25)$$

Then the action becomes

$$A = i \int d\tau e^{-\frac{\alpha}{2}} d\chi d\theta d\varphi \sin \theta f^\dagger (\chi, \theta, \varphi; \tau) \left[2(\mathrm{I} \otimes \mathrm{I}) \frac{\partial}{\partial \tau} + \right.$$

$$\left. \Sigma^\chi \left(\frac{\partial}{\partial \chi} - \frac{1}{\xi (\chi)} \right) + \frac{1}{\xi (\chi)} \left(\Sigma^\theta \frac{\partial}{\partial \theta} + \frac{1}{\sin \theta} \Sigma^\varphi \frac{\partial}{\partial \varphi} \right) \right] f (\chi, \theta, \varphi; \tau). \quad (26)$$

We define a rotation from the $\widehat{r}$ to x^3-axis. Under this rotation the spinors $f (\chi, \theta, \varphi; \tau)$ rotate as

$$f (\chi, \theta, \varphi; \tau) \to e^{i\omega \tau} S F (\chi, \theta, \varphi), \quad (27)$$

where S is the rotation operator of the spinors and defined as

$$S = e^{-\frac{i}{2} \Sigma^2 \theta - \frac{i}{2} \Sigma^1 \varphi}. \quad (28)$$

Then the action becomes

$$A = i \int d\tau e^{-\frac{\alpha}{2}} d\chi d\theta d\varphi \sin \theta F^\dagger (\chi, \theta, \varphi) \left\{ 2i\omega + \Sigma^3 \left(\frac{\partial}{\partial \chi} - \frac{1}{\xi (\chi)} \right) + \right.$$

7

$$\frac{1}{\xi\left(\chi\right)}[\frac{1}{2}\left(\Sigma^{1}+i\Sigma^{2}\right)\partial_{+}+\frac{1}{2}\left(\Sigma^{1}-i\Sigma^{2}\right)\partial_{-}-\frac{i}{2}\left[\Sigma^{1},\Sigma^{2}\right]]\}F\left(\chi,\theta,\varphi\right),\quad(29)$$

where we have used the following properties of the S operators:

$$S^{-1}\frac{\partial}{\partial\tau}S=\frac{\partial}{\partial\tau},$$

$$S^{-1}\Sigma^{\chi}\frac{\partial}{\partial\chi}S=\Sigma^{3}\frac{\partial}{\partial\chi},$$

$$S^{-1}\Sigma^{\theta}\frac{\partial}{\partial\theta}S=\Sigma^{1}\left(\frac{\partial}{\partial\theta}-i\Sigma^{2}\right),\quad(30)$$

$$S^{-1}\Sigma^{\varphi}\frac{1}{\sin\varphi}\frac{\partial}{\partial\varphi}S=\Sigma^{2}\left(\frac{1}{\sin\varphi}\frac{\partial}{\partial\varphi}+\frac{i}{2}\Sigma^{1}-\frac{i}{2}\Sigma^{3}\cot\theta\right),$$

and $\partial_{\pm}$ are

$$\partial_{\pm}=\mp\frac{\partial}{\partial\theta}+\frac{i}{\sin\theta}\frac{\partial}{\partial\varphi}+\frac{1}{2}\Sigma^{3}\cot\theta.\quad(31)$$

The $\partial_{\pm}$ are the rising and lowering operators of the angular momentum eigenfunctions

$$D^{j}_{\lambda,m}\left(\theta,\varphi\right)=\langle\lambda\left|R\left(\theta,\varphi\right)\right|jm\rangle.\quad(32)$$

We expand the spinors F in terms of the angular momentum eigenfunctions as

$$F\left(\chi,\theta,\varphi\right)=4\pi\sum_{jm}\left(2j+1\right)\begin{pmatrix}F^{jm}_{+}\left(\chi\right)D^{j}_{+1,m}\left(\theta,\varphi\right)\\F^{jm}_{\circ}\left(\chi\right)D^{j}_{\circ,m}\left(\theta,\varphi\right)\\F^{jm}_{\circ}\left(\chi\right)D^{j}_{\circ,m}\left(\theta,\varphi\right)\\F^{jm}_{-}\left(\chi\right)D^{j}_{-1,m}\left(\theta,\varphi\right)\end{pmatrix}.\quad(33)$$

Then we calculate the operation of $\partial_{\pm}$ on $F\left(\chi,\theta,\varphi\right)$ by using the following rules:

$$\partial_{\pm}D^{j}_{\lambda,m}\left(\theta,\varphi\right)=[\left(j\pm\lambda+1\right)\left(j\pm\lambda\right)]^{\frac{1}{2}}D^{j}_{\lambda\pm1,m}\left(\theta,\varphi\right).\quad(34)$$

We perform the matrix operations and the angular integrations by using the orthogonality relations of the $D^{j}_{\lambda,m}\left(\theta,\varphi\right)$ functions. The result is the following 2-dimensional form of the action:

$$A=2i\left(4\pi\right)^{3}\sum_{jm}\left(2j+1\right)\int d\tau e^{-\frac{\alpha}{2}}d\chi F^{\dagger}_{jm}\left(\chi\right)[2i\omega+$$

$$\Sigma^3 \frac{\partial}{\partial \chi} - \frac{i}{\xi(\chi)} \sqrt{j(j+1)} \Sigma^2] F_{jm}(\chi), \tag{35}$$

where $F_{jm}(\chi)$ is

$$F_{jm}(\chi) = \begin{pmatrix} F_+^{jm}(\chi) \\ F_\circ^{jm}(\chi) \\ F_\circ^{jm}(\chi) \\ F_-^{jm}(\chi) \end{pmatrix}. \tag{36}$$

The variation of the action with respect to $F_+^{jm}(\chi)$ gives the following radial equations for the photon:

$$[i\omega + \frac{1}{2}\Sigma^3 \frac{\partial}{\partial \chi} - \frac{i}{2\xi(\chi)} \sqrt{j(j+1)} \Sigma^2] F_{jm}(\chi) = 0. \tag{37}$$

By using the explicit form of the Pauli matrices we write this equation for the components of the spinor F. These are

$$\left(i\omega + \frac{\partial}{\partial \chi} \right) F_+(\chi) - \frac{1}{\xi(\chi)} \sqrt{j(j+1)} F_\circ(\chi) = 0,$$

$$\left(i\omega - \frac{\partial}{\partial \chi} \right) F_-(\chi) + \frac{1}{\xi(\chi)} \sqrt{j(j+1)} F_\circ(\chi) = 0, \tag{38}$$

$$i\omega F_\circ(\chi) - \frac{1}{2\xi(\chi)} \sqrt{j(j+1)} \left[F_-(\chi) - F_+(\chi) \right] = 0.$$

Combination of these three equations give

$$\left[\omega^2 + \frac{\partial^2}{\partial \chi^2} - \frac{j(j+1)}{\xi^2(\chi)} \right] [F_-(\chi) - F_+(\chi)] = 0. \tag{39}$$

Since $\xi(\chi) = \frac{1}{\sqrt{k}} \sin \sqrt{k}\chi$, the regular solution of this equation at the origin is given by

$$[F_-(\chi) - F_+(\chi)] = C \left(\frac{\sin \sqrt{k}\chi}{\sqrt{k}} \right)^{j+1} {}_2F_1(\alpha, \beta, \gamma; \sin^2 \sqrt{k}\chi), \tag{40}$$

where C is the normalization constant and α, β and γ are $\alpha = \frac{1}{2}\left(j+1+\frac{\omega}{k} \right)$, $\beta = \frac{1}{2}\left(j+1-\frac{\omega}{k} \right)$, $\gamma = j + \frac{3}{2}$. Then the spinor Φ becomes

$$\Phi_\pm = C \frac{e^{i\omega\tau}}{2a^2(t)} \left(\frac{\sin \sqrt{k}\chi}{\sqrt{k}} \right)^{j+1} \times$$

9

$$\{\left[1 \pm \frac{i\sqrt{k}}{\omega}(j+1)\cot\sqrt{k}\chi\right] {}_2F_1\left(\alpha,\beta,\gamma;\sin^2\sqrt{k}\chi\right) \pm$$

$$\frac{i\sqrt{k}}{2\omega}\frac{\alpha\beta}{\gamma}\sin 2\left(\sqrt{k}\chi\right) {}_2F_1\left(\alpha+1,\beta+1,\gamma+1;\sin^2\sqrt{k}\chi\right)\},$$

$$\Phi_\circ = iC\frac{e^{i\omega\tau}}{\omega a^2(t)}\sqrt{j(j+1)}\left(\frac{\sin\sqrt{k}\chi}{\sqrt{k}}\right)^j {}_2F_1(\alpha,\beta,\gamma;\sin^2\sqrt{k}\chi), \qquad (41)$$

where the conformal time τ is related to physical time t by

$$\tau = \int^t \frac{dt'}{a(t')}. \qquad (42)$$

4 Conclusion

In this paper we derived the massless particle limit of the Duffin-Kemer - Petiau equation for the spin-1 particle in the curved space and found the exact solution of this equation in the RW expanding universes. Because of the conformal and rotational symmetries of the metric we eliminated the time and angular coordinates by using the group theoretical methods and reduced the problem into the radial coordinates.

Since the particle is massless the dependence of the wave functions on the physical or parametric time do not depend on the explicit form of the expansion factor $a(t)$. Then in the current density $\Psi^\dagger\sigma\Psi$ there are no time oscillations, no particle creation for the massless particle [5].

Appendix A

Here we solve the Maxwell equations in RW metrics for the completeness. We use the metric in Eq.(24). Then the nonzero component of vierbeins are

$$L_\tau^\circ = e^{\frac{\alpha}{2}}, L_\chi^1 = e^{\frac{\alpha}{2}}, L_\theta^2 = e^{\frac{\alpha}{2}}\xi, L_\varphi^3 = e^{\frac{\alpha}{2}}\xi\sin\theta. \qquad (A1)$$

Then we can write the contravariant field strengths $F^{\mu\nu}$ in the general coordinate as

$$F^{01} = e^{-\alpha}E^1,$$

10

$$F^{02} = \frac{e^{-\alpha}}{\xi} E^2,$$

$$F^{03} = \frac{e^{-\alpha}}{\xi \sin \theta} E^3, \tag{A2}$$

$$F^{12} = \frac{e^{-\alpha}}{\xi} B^3,$$

$$F^{23} = \frac{e^{-\alpha}}{\xi^2 \sin \theta} B^1,$$

$$F^{31} = \frac{e^{-\alpha}}{\xi \sin \theta} B^2,$$

where E^i and B^i are the components of the electric and magnettic fields in the local Lorentz frame. Covariant components of the field strength tensor are

$$F_{01} = -e^\alpha E^1,$$

$$F_{02} = -e^\alpha \xi E^2, \tag{A3}$$

$$F_{03} = -e^\alpha \xi \sin \theta E^3,$$

and

$$F_{12} = e^\alpha \xi B^3,$$

$$F_{23} = e^\alpha \xi^2 \sin \theta B^1, \tag{A4}$$

$$F_{31} = e^\alpha \xi \sin \theta B^2.$$

The Maxwell equations in the free space are

$$\frac{1}{\sqrt{-g}} \left(\sqrt{-g} F^{\mu\nu} \right)_{,\nu} = 0, \tag{A5}$$

and

$$F_{\mu\nu,\sigma} + F_{\sigma\mu,\nu} + F_{\nu\sigma,\mu} = 0. \tag{A6}$$

In terms of the components these are

$$\frac{1}{\xi^2} \frac{\partial}{\partial \chi} \left[\xi^2 \left(\begin{array}{c} E^1 \\ B^1 \end{array} \right) \right] + \frac{1}{\xi \sin \theta} \frac{\partial}{\partial \theta} \left[\sin \theta \left(\begin{array}{c} E^2 \\ B^2 \end{array} \right) \right] + \frac{1}{\xi \sin \theta} \frac{\partial}{\partial \varphi} \left(\begin{array}{c} E^3 \\ B^3 \end{array} \right) = 0,$$

11

$$\frac{1}{\xi \sin \theta} \{ \frac{\partial}{\partial \theta} \left[\sin \theta \left(\begin{array}{c} E^3 \\ B^3 \end{array} \right) \right] - \frac{\partial}{\partial \varphi} \left(\begin{array}{c} E^2 \\ B^2 \end{array} \right) \} - e^{-\alpha} \frac{\partial}{\partial \tau} \left[e^\alpha \left(\begin{array}{c} -B^1 \\ E^1 \end{array} \right) \right] = 0,$$

$$\frac{1}{\xi} \{ \frac{1}{\sin \theta} \frac{\partial}{\partial \varphi} \left(\begin{array}{c} E^1 \\ B^1 \end{array} \right) - \frac{\partial}{\partial \chi} \left[\xi \left(\begin{array}{c} E^3 \\ B^3 \end{array} \right) \right] \} - e^{-\alpha} \frac{\partial}{\partial \tau} \left[e^\alpha \left(\begin{array}{c} -B^2 \\ E^2 \end{array} \right) \right] = 0, \quad (A7)$$

$$\frac{1}{\xi} \{ \frac{\partial}{\partial \chi} \left[\xi \left(\begin{array}{c} E^2 \\ B^2 \end{array} \right) \right] - \frac{\partial}{\partial \theta} \left(\begin{array}{c} E^1 \\ B^1 \end{array} \right) \} - e^{-\alpha} \frac{\partial}{\partial \tau} \left[e^\alpha \left(\begin{array}{c} -B^3 \\ E^3 \end{array} \right) \right] = 0.$$

The complex spinors η and $\eta^\dagger$ are defined as

$$\eta = \left(\begin{array}{c} E^1 + iB^1 \\ E^3 + iB^3 \\ E^2 + iB^2 \end{array} \right), \quad (A8)$$

and

$$\eta^\dagger = \left(E^1 - iB^1 \,,\, E^3 - iB^3 \,,\, E^2 - iB^2 \right). \quad (A9)$$

Then the spinor form of the Maxwell equations are

$$\frac{1}{\xi^2} \frac{\partial}{\partial \chi} \left(\xi^2 \eta^1 \right) + \frac{1}{\xi \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \, \eta^2 \right) + \frac{1}{\xi \sin \theta} \frac{\partial \eta^3}{\partial \varphi} = 0,$$

$$\frac{1}{\xi} \left[\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \, \eta^3 \right) - \frac{1}{\sin \theta} \frac{\partial \eta^2}{\partial \varphi} \right] - ie^{-\alpha} \frac{\partial}{\partial \tau} \left(e^\alpha \eta^1 \right) = 0,$$

$$\frac{1}{\xi} \left[\frac{1}{\sin \theta} \frac{\partial \eta^1}{\partial \varphi} - \frac{\partial}{\partial \chi} \left(\xi \eta^3 \right) \right] - ie^{-\alpha} \frac{\partial}{\partial \tau} \left(e^\alpha \eta^2 \right) = 0 \quad (A10)$$

$$\frac{1}{\xi} \left[\frac{\partial}{\partial \chi} \left(\xi \, \eta^2 \right) - \frac{\partial \eta^1}{\partial \theta} \right] - ie^{-\alpha} \frac{\partial}{\partial \tau} \left(e^\alpha \eta^3 \right) = 0.$$

We define the following new components as

$$\eta^\pm = \frac{1}{2} \left(\eta^1 \mp \eta^2 \right) = \left(\begin{array}{c} \eta_+^{jm} (\chi; \tau) \, D_{+1,m}^j (\theta, \varphi) \\ \eta_-^{jm} (\chi; \tau) \, D_{-1,m}^j (\theta, \varphi) \end{array} \right), \quad (A11)$$

and

$$\eta^3 = 4\pi \sum_{jm} (2j + 1) \, \eta_0^{jm} (\chi, \tau) \, D_{0,m}^j (\theta, \varphi). \quad (A12)$$

12

Then the spinor η becomes

$$\eta\left(\chi,\theta,\varphi;\tau\right) = 4\pi \sum_{jm} \left(2j+1\right)$$

$$\times \begin{pmatrix} [\eta_+^{jm}\left(\chi;\tau\right) D_{+1,m}^j\left(\theta,\varphi\right) + \eta_-^{jm}\left(\chi;\tau\right) D_{-1,m}^j\left(\theta,\varphi\right)] \\ \eta_0^{jm}\left(\chi,\tau\right) D_{0,m}^j\left(\theta,\varphi\right) \\ \frac{1}{i}[\left(\eta_+^{jm}\left(\chi;\tau\right) D_{+1,m}^j\left(\theta,\varphi\right) - \eta_-^{jm}\left(\chi;\tau\right) D_{-1,m}^j\left(\theta,\varphi\right)\right)] \end{pmatrix}. \qquad (A13)$$

We substitute the expansion into equation and $\partial_\pm D$ properties of the D functions: Then we obtain the following radial equations:

$$\frac{\sqrt{j\left(j+1\right)}}{\xi}\left(\eta_+^{jm} + \eta_-^{jm}\right) = \frac{\partial}{\partial\tau}\left[e^{-\alpha}\left(e^\alpha\eta_0^{jm}\right)\right],$$

$$\frac{1}{\xi^2}\frac{\partial}{\partial\chi}\left(\xi^2\eta_0^{jm}\right) = \frac{\sqrt{j\left(j+1\right)}}{\xi}\left(\eta_+^{jm} - \eta_-^{jm}\right),$$

$$\frac{1}{\xi}\frac{\partial}{\partial\chi}\left[\xi\left(\eta_+^{jm} + \eta_-^{jm}\right)\right] = e^{-\alpha}\frac{\partial}{\partial\tau}\left[e^\alpha\left(\eta_+^{jm} - \eta_-^{jm}\right)\right], \qquad (A14)$$

$$\frac{\sqrt{j\left(j+1\right)}}{\xi}\eta_0^{jm} - \frac{1}{\xi}\frac{\partial}{\partial\chi}\left[\xi\left(\eta_+^{jm} - \eta_-^{jm}\right)\right] = e^{-\alpha}\frac{\partial}{\partial\tau}\left[e^\alpha\left(\eta_+^{jm} + \eta_-^{jm}\right)\right].$$

The combination of these equations give the following wave equation in $1+1$ dimensions:

$$\left(\frac{\partial^2}{\partial\chi^2} - \frac{j\left(j+1\right)}{\xi^2} - \frac{\partial^2}{\partial\tau^2}\right)\left(e^\alpha\xi^2\eta_0\right) = 0. \qquad (A15)$$

This is the same with Eqs.(39) and the solutions of it the same with Eqs.(40) and Eqs.(41).

References

[1] E. A. Power and J. A. Wheeler, *Rev. Mod. Phys.* **29**, 480 (1957); D. R. Brill and J. A. Wheeler, *Rev. Mod. Phys.* **29**, 465 (1957).

[2] A. O. Barut and I. H. Duru, *Phys. Rev.* **A36**, 3705 (1987).

[3] A. O. Barut and L. P. Singh, *Comm. Theo. Phys.*,**2**, 1 (1993).

[4] V. M. Villalba and U. Percoco, *J. Math. Phys.*, **31**, 3 (1990).

[5] N. G. Sarkar and S. Biswas, *Int.J.Mod.Phys.*, **A15**, 497 (2000).

[6] G. P. Oppenheimer, *Phys.Rev.* **38**,725 (1931).

[7] R. H. Good Jr. *Phys.Rev.* **105**, 1914 (1957); T.J. Nelson and R. H. Good, Jr. *Phys.Rev.* **179**, 1945 (1969).

[8] S. Weinberg, *Phys.Rev.* **134** B882 (1964).

[9] E. Recami et al., *Lett.Nouvo Cim.* **11**, 568 (1974); E. Gianatto, *Lett.Nouvo Cim.* **44**, 140 (1985).

[10] V. V. Dvoeglazov, Yu. N. Tyukhtyaev and Kyudyakov, Izv. VUZ: *Fiz.* **37**, 110 (1994) [English translation: *Russ.Phys.J.* **37**, 898 (1994)].

[11] A. Gersten, Found.*Phys. Lett.*, **12**, 291 (1998).

[12] D. H. Kobe, *Phys. Lett.* A **253**, 7 (1999).

[13] U. Leonhardt and P. Piwnicki, *Phys. Rev. Lett.* **84**, 822 (2000).

[14] S. A. Hugget and K. P. Tod, *An Introduction to Twistor Theory*, (Cambridge University Press 1994).

[15] H. C. Corben, *Classical and Quantum Dynamics of Spinning Test Particles*, (Holden Day, NY, 1968); H. A. Kramers, *Quantum Mechanics*, (Dover NY, 1964); A. T. Ogelski and J. Sobczk, *J. Math. Phys.*, **22**, 2060 (1981).

[16] A. O. Barut and N. Zanghi, *Phys. Rev. Lett.*, **52**, 2009 (1984).

[17] A. O. Barut and M. Pavsic, *Class. Quantum Gravit.*, **A**L137 (1987); A. O. Barut and I. H. Duru, *Phys. Rev. Lett.*, **53**, 2355 (1984) and *Phys. Rep.* **C172**, 1 (1989).

[18] A. O. Barut, *Phys. Lett. B*, **237**, 436 (1990); N. Ünal, *Found. Phys.*, **27**, 811 (1997).

[19] N. Ünal, *Found. Phys.*, **27**, 795(1997).

[20] A. O. Barut and N. Ünal, *Forts. Phys.*, **33**, 319 (1985).